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Deep Doubly Curved Multilayered Shell Theory

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A two-dimensional theory is presented for the analysis of deep, doubly curved, multilayered shells. The theory is based on a kinematical approach in which the continuity conditions for displacements and shear stresses at layer interfaces and on the bounding surfaces of the shell are exactly satisfied. It also takes into account refinements of the shear and membrane terms, by means of trigonometric functions as proposed previously for the transverse shear. The accuracy of the proposed theory is assessed through investigation of significant problems for which an exact three-dimensional elasticity solution is known: first, the bending of a three-layered, laminated, symmetric cross-ply rectangular plate, simply supported along all edges, submitted to a double sinusoidal transverse loading, and second, the bending of a circular, cylindrical panel. Results obtained with the model are compared with those yielded by previous theories. The sensitivity of the model to edge effects, for a hard-clamped, free-edge cylindrical panel, is also examined.

I. Introduction

DEEP shells of laminated composite are being increasingly used in structural applications, and require the development of efficient tools for their design and sizing. Theories that account for interlaminar stresses are not numerous. One of the most used ways is to introduce piecewise linear approximation for displacements, as Di Sciuva,[1] who assumed the in-plane displacements to be piecewise linear functions of the thickness coordinate and imposed the continuity of the transverse shear stresses at layer interfaces. Such theories, however, do not allow the bounding conditions for transverse shear stresses on the top and bottom surfaces of the shell to be satisfied. Timarci and Soldatos[2] introduced arbitrary shear-deformation shape functions, thus leaving open possibilities for particular shear-deformable shell theories, such as the parabolic shear-deformable shell theory, which allows the boundary conditions on the top and bottom surfaces of the shell to be satisfied. Recently, He[3] proposed a refined third-order model that allowed both the compatibility conditions at layer interfaces and the bounding conditions for the transverse shear stresses to be satisfied. A third-order theory allowing the satisfaction of those latter conditions was also proposed by Shu.[4]

This paper presents a new approach for developing a simple and refined theory for deep, doubly curved, laminated shells, in which the continuity conditions for displacements and transverse shear stresses at layer interfaces and on the bounding surfaces of the shell are exactly satisfied. The theory contains the same five generalized displacements as the shear-deformation theory and is based on a new assumed displacement field in which the shear is represented by a sine function and where the membrane terms are refined by a cosine function. The introduction of trigonometric functions in the kinematics, in place of polynomial development of the thickness coordinate, was first suggested by Touratier.[5] It then was used by Stein[6] to represent the transverse shear, but the Stein theory did not allow the boundary conditions for the shear stresses on the top and bottom surfaces of the shell to be satisfied. It was used again by Touratier,[7] but only in the case of shallow shells. The Touratier model has since been extended to multilayered shallow shells by Touratier and Béakou[8] and Béakou and Touratier.[9] However, that proposed model cannot be applied to deep shells.

The proposed model extends the work of Refs. 7 and 9–11 to moderately thick deep laminated shells. In addition to the Touratier model, the contact problem for shear stresses at layer interfaces is solved and refinements of the membrane terms are taken into account, by means of a cosine function.

Note that, developing the sine and cosine functions to various orders, previous plate theories[12–14] can be obtained. The model is evaluated through significant problems, for which an exact three-dimensional elasticity solution is known. Comparison with the results obtained by previous theories show the improvement of the model. It is also shown that, unlike some shear-deformation theories, no shear-correction factors are needed. The sensitivity of the model to edge effects is also tested on a particular case.

II. Kinematical Shell Model with Interlayer Continuity

A. Introduction to Shell Theory

We consider an undeformed laminated shell of constant thickness h, consisting of an arrangement of a finite number N of orthotropic layers (see Fig. 1). The space occupied by the shell is V. The boundary of the shell is the reunion of the upper surface Ω_h, the lower surface Ω_0, and the edge faces A.

The interface between the ith and $(i+1)$th layer is Ω_i, the distance between Ω_0 and Ω_i is $x_{3(i)}$. The reference surface coincides with the bottom surface of the shell Ω_0. In this paper, the Einsteinian summation convention applies to repeated indices, where Latin indices range from 1 to 3 and Greek indices range from 1 to 2.

A point M out of the reference surface being given, let us denote P the point of the reference surface closest to M. Covariant base vectors $\boldsymbol{a}_i$ and $\boldsymbol{g}_i$ and contravariant base vectors $\boldsymbol{a}^i$ and $\boldsymbol{g}^i$ in the undeformed state of the shell are introduced as

$$\boldsymbol{a}_\alpha = P_{,\alpha}, \qquad \boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2\|}, \qquad (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)\cdot \boldsymbol{a}_3 > 0$$
$$\boldsymbol{g}_i = M_{,i}, \qquad (\boldsymbol{g}_1 \wedge \boldsymbol{g}_2)\cdot \boldsymbol{g}_3 > 0, \qquad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta \qquad (1)$$
$$\boldsymbol{a}^3 = \boldsymbol{a}_3, \qquad \boldsymbol{g}^\alpha \cdot \boldsymbol{g}_\beta = \delta^\alpha_\beta, \qquad \boldsymbol{g}^3 = \boldsymbol{g}_3$$

and differentiation with respect to x_i is denoted by $,i$ with $[\delta^\beta_\alpha]$ the identity tensor. Recall that

$$M = P + x_3 \boldsymbol{a}^3 \qquad (2)$$

Equations (1) and (2) ensure the following relations (see, for instance, Ref. 13):

$$\boldsymbol{g}_\alpha = \mu^\beta_\alpha \boldsymbol{a}_\beta, \qquad \boldsymbol{g}_3 = \boldsymbol{a}_3, \qquad \boldsymbol{g}^\alpha = -\mu^{\alpha}_{\beta^{-1}} \boldsymbol{a}^\beta$$
$$\boldsymbol{g}^3 = \boldsymbol{a}^3, \qquad \boldsymbol{g}_\alpha = g_{\alpha\beta}\boldsymbol{g}^\beta, \qquad \boldsymbol{g}^\alpha = g^{\alpha\beta}\boldsymbol{g}_\beta \qquad (3)$$
$$\boldsymbol{a}_\alpha = a_{\alpha\beta}\boldsymbol{a}^\beta, \qquad \boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta$$

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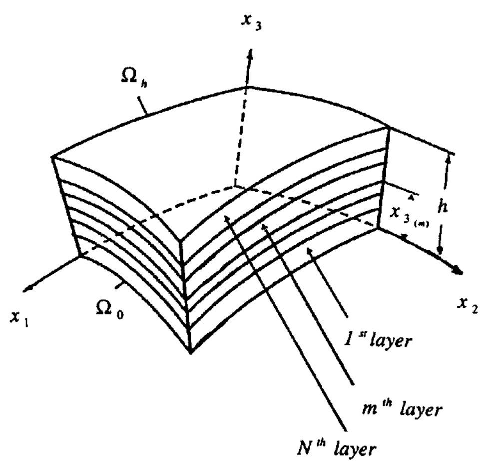

Fig. 1 Geometry of the laminated shell.

in which the components of the shifter tensor are

$$\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - b^{\alpha}_{\beta} x_3 \tag{4}$$

those of the curvature tensor are

$$b_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}^3 \tag{5}$$

and its mix components are

$$b^{\alpha}_{\beta} = -\boldsymbol{a}_{3,\beta} \cdot \boldsymbol{a}^{\alpha} \tag{6}$$

The surface metrics α_1 and α_2 are related to the $a_{\alpha\beta}$ coefficients via

$$\alpha_l^2 = a_{ll} \tag{7}$$

(no summation on l index).

In the following, the curvilinear coordinates (or shell coordinates) are assumed orthogonal and are such that the x_1 and x_2 curves are lines of curvature on the reference surface $x_3 = 0$; x_3 curves are straight lines perpendicular to the surface $x_3 = 0$. R_1 and R_2 are the values of the principal radii of curvature of the reference surface.

The distance ds between two points $P(x_1, x_2, 0)$ and $P'(x_1 + dx_1, x_2 + dx_2, 0)$ of the reference surface Ω_0 of the shell is given by

$$(ds)^2 = \alpha_1^2 (dx_1)^2 + \alpha_2^2 (dx_2)^2 \tag{8}$$

where α_1 and α_2 are the surface metrics

$$\alpha_l^2 = \left(\frac{\partial P}{\partial x_l}\right)\left(\frac{\partial P}{\partial x_l}\right) \tag{9}$$

The distance dS between two points $M(x_1, x_2, x_3)$ and $M'(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ out of the reference surface is given by

$$(dS)^2 = L_1^2 (dx_1)^2 + L_2^2 (dx_2)^2 + L_3^2 (dx_3)^2 \tag{10}$$

where L_1, L_2, and L_3 are the so-called Lamé coefficients

$$L_1 = \alpha_1[1 + (x_3/R_1)], \qquad L_2 = \alpha_2[1 + (x_3/R_2)], \qquad L_3 = 1 \tag{11}$$

B. Kinematic Assumptions

Geometrical linear shells are considered, including an elastic-linear behavior for laminates. The components of the displacement field of any point $M(x_1, x_2, x_3)$ of the volume occupied by the shell, V, expressed in the contravariant basis $(\boldsymbol{g}^{\alpha}, \boldsymbol{g}^3)$ are assumed to be in the following form:

$$U_\alpha = u_\alpha + x_3 \eta_\alpha + f(x_3)\gamma_\alpha^0 + g(x_3)\varphi_\alpha + \sum_{m=1}^{N-1} u_{(m)\alpha}\left(x_3 - x_{3_{(m)}}\right) H\left(x_3 - x_{3_{(m)}}\right) \tag{12}$$

$$U_3 = w$$

where [as suggested by Touratier[7] for $f(x_3)$]

$$f(x_3) = (h/\pi)\sin(\pi x_3/h), \qquad g(x_3) = (h/\pi)\cos(\pi x_3/h) \tag{13}$$

where H is the Heaviside step function defined by

$$H\left(x_3 - x_{3_{(m)}}\right) = \begin{cases} 1 & \text{for} \quad x_3 \geq x_{3_{(m)}} \\ 0 & \text{for} \quad x_3 < x_{3_{(m)}} \end{cases} \tag{14}$$

This step function has been previously used by Di Sciuva[1] and He,[3] among others. Also, as in Ref. 7, the choice for $f(x_3)$ can be justified in a discrete-layer approach from the three-dimensional works of Cheng[15] for thick plates.

In this displacement field, u_α are membrane displacements, γ_α^0 are the transverse shear strains at $x_3 = 0$, and w is the transverse deflection of the shell.

The $g(x_3)\varphi_\alpha$ terms are refinements of membrane displacements, and η_α and φ_α are functions to be determined using the boundary conditions for the transverse shear stresses on the top and bottom surfaces of the shell. With the help of the $u_{(m)\alpha}$, which represent the generalized displacements per layer, the continuity of the displacements at layer interfaces are automatically satisfied from the Heaviside function. They are determined from the continuity conditions on the transverse shear stresses at the interfaces. Moreover, keeping γ_α^0 functions in Eq. (12) allows finding Mindlin's theory for homogeneous plates either by developing by means of Taylor expansion the sine at the first order or by setting $f(x_3) = x_3$.

To reduce the number of unknowns in the displacement field, the continuity conditions of transverse shear stresses at layer interfaces and on the bounding surfaces will be used in the following. The transverse normal stress is ignored, and it is assumed that no tangential tractions are exerted on the upper and lower surfaces of the shell.

III. Interface Continuity and Boundary Conditions

A. Linear Constitutive Law

Recall that the coefficients of the constitutive law are generally given in a coordinate system related to the material (material coordinates), whereas we presently deal with shell coordinates.

Taking into account the nullity of the transverse normal stress, the orthotropic constitutive law per ith layer can be written as follows, in material coordinates:

$$\sigma_{\alpha\alpha}^{(i)^{\text{mat}}} = C_{\alpha\alpha\beta\beta}^{(i)^{\text{mat}}} e_{\beta\beta}^{\text{mat}}, \qquad \sigma_{\alpha\beta}^{(i)^{\text{mat}}} = C_{\alpha\beta\alpha\beta}^{(i)^{\text{mat}}} e_{\alpha\beta}^{\text{mat}} \qquad (\beta \neq \alpha)$$

$$\sigma_{\alpha 3}^{(i)^{\text{mat}}} = C_{\alpha 3\alpha 3}^{(i)^{\text{mat}}} e_{\alpha 3}^{\text{mat}} \tag{15}$$

or in a matrix form as

$$\left\{\sigma^{(i)^{\text{mat}}}\right\} = \left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]\{\boldsymbol{e}^{\text{mat}}\} \tag{16}$$

where

$$\left\{\sigma^{(i)^{\text{mat}}}\right\} = \left\{\sigma_{11}^{(i)^{\text{mat}}}, \sigma_{22}^{(i)^{\text{mat}}}, \sigma_{31}^{(i)^{\text{mat}}}, \sigma_{12}^{(i)^{\text{mat}}}, \sigma_{32}^{(i)^{\text{mat}}}\right\} \tag{17}$$

and

$$\{\boldsymbol{e}^{\text{mat}}\} = \left\{e_{11}^{\text{mat}}, e_{22}^{\text{mat}}, e_{31}^{\text{mat}}, e_{12}^{\text{mat}}, e_{32}^{\text{mat}}\right\} \tag{18}$$

are the stress and strain vectors, respectively.

The i exponent refers to the ith layer and the mat exponent refers to the material coordinates. Note that the $C_{\alpha\beta\gamma\delta}^{(i)^{\text{mat}}}$ coefficients are bidimensional coefficients, related to the three-dimensional $C_{\alpha\beta\gamma\delta}^{(i)^{\text{3Dmat}}}$ ones via

$$C_{\alpha\alpha\beta\beta}^{(i)^{\text{mat}}} = C_{\alpha\alpha\beta\beta}^{(i)^{\text{3Dmat}}} - \frac{C_{\alpha\alpha 33}^{(i)^{\text{3Dmat}}} C_{\beta\beta 33}^{(i)^{\text{3Dmat}}}}{C_{3333}^{(i)^{\text{3Dmat}}}}$$

$$C_{\alpha\beta\alpha\beta}^{(i)^{\text{mat}}} = C_{\alpha\beta\alpha\beta}^{(i)^{\text{3Dmat}}} \qquad (\beta \neq \alpha) \tag{19}$$

$$C_{\alpha\alpha 33}^{(i)^{\text{mat}}} = C_{\alpha\alpha 33}^{(i)^{\text{3Dmat}}}$$

If $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ is the material Cartesian system related to the layers, a point M of the structure will be located by its coordinates (X_1, X_2, X_3) in this system, which are functions of the shell coordinates x_α.

The covariant vectors and material vectors are related via

$$\boldsymbol{g}_\alpha = \frac{\partial \boldsymbol{M}}{\partial x_\alpha} = X_{\beta,\alpha}\boldsymbol{E}_\beta, \qquad \boldsymbol{g}_3 = \boldsymbol{E}_3 \tag{20}$$

Let $[\boldsymbol{T}]$ be the matrix defined by

$$\begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \boldsymbol{g}_3 \end{bmatrix} = [\boldsymbol{T}] \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \\ \boldsymbol{E}_3 \end{bmatrix} \tag{21}$$

the components of which are given by

$$T_{\alpha\beta} = X_{\alpha,\beta}, \qquad T_{i3} = \delta_{i3} \tag{22}$$

Set

$$\left\{\sigma^{(i)^{\text{mat}}}\right\}_{\text{plane}} = \begin{Bmatrix} \sigma_{11}^{(i)^{\text{mat}}} \\ \sigma_{22}^{(i)^{\text{mat}}} \\ \sigma_{12}^{(i)^{\text{mat}}} \end{Bmatrix}, \qquad \left\{\sigma^{(i)}\right\}_{\text{plane}} = \begin{Bmatrix} \sigma_{11}^{(i)} \\ \sigma_{22}^{(i)} \\ \sigma_{12}^{(i)} \end{Bmatrix}$$
$$\{\boldsymbol{e}^{\text{mat}}\}_{\text{plane}} = \begin{Bmatrix} e_{11}^{\text{mat}} \\ e_{22}^{\text{mat}} \\ e_{12}^{\text{mat}} \end{Bmatrix}, \qquad \{\boldsymbol{e}\}_{\text{plane}} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{Bmatrix} \tag{23}$$

$$\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{plane}} = \begin{bmatrix} C_{1111}^{(i)^{\text{mat}}} & C_{1122}^{(i)^{\text{mat}}} & 0 \\ C_{2211}^{(i)^{\text{mat}}} & C_{2222}^{(i)^{\text{mat}}} & 0 \\ 0 & 0 & 2C_{1212}^{(i)^{\text{mat}}} \end{bmatrix}$$
$$\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{shear}} = \begin{bmatrix} 2C_{1313}^{(i)^{\text{mat}}} & 2C_{1323}^{(i)^{\text{mat}}} \\ 2C_{1323}^{(i)^{\text{mat}}} & 2C_{2323}^{(i)^{\text{mat}}} \end{bmatrix} \tag{24}$$

Then we have, in shell coordinates for each layer

$$\left\{\sigma^{(i)}\right\}_{\text{plane}} = [\boldsymbol{T}]\left\{\sigma^{(i)^{\text{mat}}}\right\}_{\text{plane}} = [\boldsymbol{T}]\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{plane}}[\boldsymbol{T}]^{-1}\{e\}_{\text{plane}}$$
$$\begin{bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{bmatrix} = [\boldsymbol{T}_{\alpha\beta}] \begin{bmatrix} \sigma_{13}^{(i)^{\text{mat}}} \\ \sigma_{23}^{(i)^{\text{mat}}} \end{bmatrix} = [\boldsymbol{T}_{\alpha\beta}]\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{shear}}[\boldsymbol{T}_{\alpha\beta}]^{-1} \begin{bmatrix} e_{13}^{(i)} \\ e_{23}^{(i)} \end{bmatrix} \tag{25}$$

or

$$\left\{\sigma^{(i)}\right\}_{\text{plane}} = \left[\boldsymbol{C}^{(i)}\right]_{\text{plane}}\{e\}_{\text{plane}}, \qquad \begin{bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{bmatrix} = \left[\boldsymbol{C}^{(i)}\right]_{\text{shear}} \begin{bmatrix} e_{13}^{(i)} \\ e_{23}^{(i)} \end{bmatrix} \tag{26}$$

where

$$\left[\boldsymbol{C}^{(i)}\right]_{\text{plane}} = [\boldsymbol{T}]\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{plane}}[\boldsymbol{T}]^{-1}$$
$$\left[\boldsymbol{C}^{(i)}\right]_{\text{shear}} = [\boldsymbol{T}_{\alpha\beta}]\left[\boldsymbol{C}^{(i)^{\text{mat}}}\right]_{\text{shear}}[\boldsymbol{T}_{\alpha\beta}]^{-1} \tag{27}$$

B. Interface and Free-Traction Boundary Conditions

The covariant shear-strain components of the shell can be obtained by the formulas[13]

$$e_{\alpha 3} = \tfrac{1}{2}\left[U_{\alpha,3} + U_{3|\alpha} + b_\alpha^\beta(U_\beta - x_3 U_{\beta,3})\right] \tag{28}$$

where $|\alpha$ is the covariant derivative on the reference surface Ω_0 with respect to x_α.

Thus, we have

$$e_{\alpha 3} = \frac{1}{2}\Bigg[\eta_\alpha + \left[\delta_\alpha^\beta f' + b_\alpha^\beta(f - x_3 f')\right]\gamma_\beta^0 + \left[\delta_\alpha^\beta g' + b_\alpha^\beta(g - x_3 g')\right]\varphi_\beta + w_{|\alpha} + b_\alpha^\beta u_\beta + \sum_{m=1}^{N-1}\left[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta\right]u_{(m)\beta} H\left(x_3 - x_{3_{(m)}}\right)\Bigg] \tag{29}$$

1. Free-Traction Boundary Conditions for the Shear at Top and Bottom Surfaces of the Shell

According to Eq. (26), the free-traction boundary conditions on the top and bottom surfaces of the shell can be written as follows:

$$e_{\alpha 3}(x_3 = 0) = 0 \tag{30a}$$

$$e_{\alpha 3}(x_3 = h) = 0 \tag{30b}$$

which yields

$$\eta_\alpha + \gamma_\alpha^0 + w_{|\alpha} + b_\alpha^\beta[u_\beta + (h/\pi)\varphi_\beta] = 0 \tag{31a}$$

and

$$\eta_\alpha + \left[-\delta_\alpha^\beta + h b_\alpha^\beta\right]\gamma_\beta^0 - b_\alpha^\beta\varphi_\beta + w_{|\alpha} + b_\alpha^\beta u_\beta + \sum_{m=1}^{N-1}\left[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta\right]u_{(m)\beta} = 0 \tag{31b}$$

Substituting η_α in Eq. (31b) by their expression from Eq. (31a), we have

$$b_\alpha^\beta\varphi_\beta = \frac{\pi}{2h}\left[-2\delta_\alpha^\beta + h b_\alpha^\beta\right]\gamma_\beta^0 + \sum_{m=1}^{N-1}\frac{\pi}{2h}\left[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta\right]u_{(m)\beta} \tag{32}$$

or

$$\varphi_\alpha = d_\alpha^\beta\gamma_\beta^0 + \sum_{m=1}^{N-1}\frac{\pi}{2h}\left[b_\alpha^\beta\right]^{-1}\left[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta\right]u_{(m)\beta} \tag{33}$$

where the tensor $[d_\alpha^\beta]$ is given by

$$\left[d_\alpha^\beta\right] = (\pi/2h)\left[b_\alpha^\beta\right]^{-1}\left[-2\delta_\alpha^\beta + h b_\alpha^\beta\right] = (\pi/2h)\left[-2\left[b_\alpha^\beta\right]^{-1} + h\left[\delta_\alpha^\beta\right]\right] \tag{34}$$

If D is the determinant of the latter system, we have

$$\varphi_\alpha = d_\alpha^\beta\gamma_\beta^0 + \sum_{m=1}^{N-1} f_{(m)^\alpha}^\beta u_{(m)\beta} \tag{35}$$

where

$$f_{(m)^\alpha}^\beta = \frac{\pi}{2hD}\left[\sum_{m=1}^{N-1}\left(\delta_\nu^\lambda - x_{3_{(m)}} b_\nu^\lambda\right)\right]\Delta_{\nu\alpha}\varepsilon_{\lambda\mu} b_\mu^\beta \tag{36}$$

and where the $\varepsilon_{\lambda\mu}$ coefficients are defined by

$$\varepsilon_{11} = \varepsilon_{22} = 0, \qquad \varepsilon_{12} = -\varepsilon_{21} = 1 \tag{37}$$

and the coefficients $\Delta_{\nu\alpha}$ by

$$\Delta_{\nu\alpha} = 1 - \delta_\alpha^\nu \tag{38}$$

Finally, the transverse shear strains can be written in the following form:

$$e_{\alpha 3} = \frac{1}{2}\Bigg\{\left[\delta_\alpha^\beta(f' - 1) + b_\alpha^\beta(f - x_3 f')\right]\gamma_\beta^0 + \left[\delta_\alpha^\beta g' + b_\alpha^\beta\left(g - \frac{h}{\pi} - x_3 g'\right)\right]\varphi_\beta + \sum_{m=1}^{N-1}\left[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta\right]u_{(m)\beta} H\left(x_3 - x_{3_{(m)}}\right)\Bigg\} \tag{39}$$

2. *Interface Conditions for the Transverse Shear Stresses*

These conditions can be written as follows:

$$\sigma_{\alpha 3}^{(i)}\left(x_3 = x_{3(i)}\right) = \sigma_{\alpha 3}^{(i+1)}\left(x_3 = x_{3(i)}\right)$$

$$(\alpha = 1, 2;\ i = 1, \ldots, N-1) \quad (40)$$

or

$$2C_{\alpha 3\omega 3}^{(i)}\left\{\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} e_{\omega 3}\left(x_{3(i)} - \varepsilon\right)\right\} = 2C_{\alpha 3\omega 3}^{(i+1)}\left\{\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} e_{\omega 3}\left(x_{3(i)} + \varepsilon\right)\right\}$$

$$(\alpha = 1, 2;\ i = 1, \ldots, N-1) \quad (41)$$

Substituting Eqs. (35) into Eq. (39) and using Eq. (41) yield

$$\left(C_{\alpha 3\omega 3}^{(i)} - C_{\alpha 3\omega 3}^{(i+1)}\right)\Bigg[\Bigg[\delta_\omega^\beta\left(f'\left(x_{3(i)}\right) - 1\right) + b_\omega^\beta\Big(f\left(x_{3(i)}\right) - x_{3(i)} f'\left(x_{3(i)}\right)\Big) + d_\nu^\beta\left[\delta_\omega^\nu g'\left(x_{3(i)}\right) + b_\omega^\nu\left(g\left(x_{3(i)}\right) - \frac{h}{\pi} - x_{3(i)} g'\left(x_{3(i)}\right)\right)\right]\Bigg]\gamma_\beta^0 + \sum_{m=1}^{i-1}\left[\left(\delta_\omega^\beta - x_{3(m)} b_\omega^\beta\right)\right]u_{(m)\beta}$$
$$+ \sum_{m=1}^{N-1}\left[\left(\delta_\omega^\nu g'\left(x_{3(i)}\right) + b_\omega^\nu\left(g\left(x_{3(i)}\right) - \frac{h}{\pi} - x_{3(i)} g'\left(x_{3(i)}\right)\right)\right) f_{(m)\nu}^\beta\right]u_{(m)\beta}\Bigg]$$
$$- C_{\alpha 3\omega 3}^{(i+1)}\left(\delta_\omega^\beta - x_{3(m)} b_\omega^\beta\right)u_{(i)\beta} = 0 \quad (42)$$

which can be regarded as a linear algebraic system of $2(N-1)$ equations for the $2(N-1)$ generalized displacements per layer $u_{(m)\alpha}(\alpha = 1, 2)$. The latter can thus be expressed as functions of the generalized displacements γ_α^0 as

$$u_{(m)\alpha} = a_{(m)\alpha}^\beta \gamma_\beta^0 \quad (43)$$

where the $a_{(m)\alpha}^\beta$ coefficients depend only on the curvatures and on the material properties of various layers. For any specified laminated shell where the b_α^β are given, all of the $a_{(m)\alpha}^\beta$ are, therefore, known constants.

C. Final Form of the Displacement Field

Combining Eq. (31a) with Eqs. (35) and (43) leads to

$$\eta_\alpha = -b_\alpha^\beta u_\beta + \Lambda_{(\gamma)\alpha}^{(\eta)\beta}\gamma_\beta^0 - w_{|\alpha} \quad (44)$$

where

$$\Lambda_{(\gamma)\alpha}^{(\eta)\beta} = -\left[\delta_\alpha^\beta + \frac{h}{\pi} b_\alpha^\lambda d_\lambda^\beta + \sum_{m=1}^{N-1}\frac{h}{\pi} f_{(m)\lambda}^\gamma b_\alpha^\lambda a_{(m)\gamma}^\beta\right] \quad (45)$$

Combining Eq. (35) with Eq. (43) yields

$$\varphi_\alpha = \Lambda_{(\gamma)\alpha}^{(\varphi)\beta}\gamma_\beta^0 \quad (46)$$

where

$$\Lambda_{(\gamma)\alpha}^{(\varphi)\beta} = -\left[\delta_\alpha^\beta + \frac{h}{\pi} b_\alpha^\lambda d_\lambda^\beta + \sum_{m=1}^{N-1}\frac{h}{\pi} f_{(m)\lambda}^\gamma b_\alpha^\lambda a_{(m)\gamma}^\beta\right] \quad (47)$$

The expressions of the displacement components thus become

$$U_\alpha = \mu_\alpha^\beta u_\beta - x_3 w_{|\alpha} + h_\alpha^\beta \gamma_\beta^0, \qquad U_3 = w \quad (48)$$

where h_α^β are known functions of x_3 defined by

$$h_\alpha^\beta = \delta_\alpha^\beta f(x_3) + x_3 \Lambda_{(\gamma)\alpha}^{(\eta)\beta} + g(x_3)\Lambda_{(\gamma)\alpha}^{(\varphi)\beta} + \sum_{m=1}^{N-1} a_{(m)\alpha}^\beta\left(x_3 - x_{3(m)}\right)H\left(x_3 - x_{3(m)}\right) \quad (49)$$

D. Justification of the Representation of Transverse Shear

In the proposed theory, the transverse shear is represented by means of the shear function $f(x_3)$ and the membrane terms by means of the function $g(x_3)$. Note that in the case of homogeneous shells, terms including the Heaviside step function H vanish and that setting $f(x_3) = x_3$ and $g(x_3) = 0$ produces the uniform shear strain

$$e_{\alpha 3} = \tfrac{1}{2}\tilde{\gamma}_\alpha^0 \quad (50)$$

where the $\tilde{\gamma}_\alpha^0$ functions, obtained by shifting the γ_α^0 functions on the midsurface of the shell, are the measure of the transverse shear on the midsurface of the shell

$$\tilde{\gamma}_\alpha^0 = \gamma_\alpha^0 - (h/2) b_\alpha^\beta \gamma_\beta^0 \quad (51)$$

The shear strain can then be related to the shear strain of the Naghdi[13] theory, measured on the midsurface of the shell

$$e_{\alpha 3}^{\text{Naghdi}} = \tfrac{1}{2}\left[\tilde{\omega}_\alpha + w_{|\alpha} + b_\alpha^\beta \tilde{u}_\beta\right] \quad (52)$$

where $\tilde{u}_\alpha$ are membrane displacements and $\tilde{\omega}_\alpha$ rotations of the undeformed normals.

Also note, in the same way, the Koiter theory[14] is obtained by setting $f(x_3) = g(x_3) = 0$ and $\gamma_\alpha^0 = 0$. The expression given in Eq. (48) for the displacement field then becomes

$$U_\alpha = \mu_\alpha^\beta u_\beta - x_3 w_{|\alpha}, \qquad U_3 = w \quad (53)$$

IV. Two-Dimensional Boundary-Value Problem

The equations of motion and the natural boundary conditions are derived via Hamilton's principle

$$\int_0^t \left\{\int_V \sigma^{ij}\delta e_{ij}\,\mathrm{d}V - \int_V \rho\ddot{\boldsymbol{U}}\cdot\delta\boldsymbol{U}\,\mathrm{d}V + \int_V \boldsymbol{f}\cdot\delta\boldsymbol{U}\,\mathrm{d}V + \int_A \boldsymbol{s}\cdot\delta\boldsymbol{U}\,\mathrm{d}A + \int_{\Omega_0}\left(-\mu_{(h)}p_h + p_0\right)\mathrm{d}S\right\}\mathrm{d}t = 0 \quad (54)$$

where $\mu_{(h)}$ is the value at $x_3 = h$ of

$$\mu = \det\left[\mu_\alpha^\beta\right] \quad (55)$$

and where a superposed dot is used for differentiation with respect to time t, ρ is the mass density, δ is the variational operator, f^i are components of body forces, s^i are the prescribed components of stress vector per unit area of lateral surface of the shell, and p_0 and p_h are the prescribed components of stress vector per unit area of the surfaces Ω_0 and Ω_h.

Performing numerical integration through the thickness of the shell, the following equations of motion are deduced from Eq. (54):

$$M_{|\beta}^{(1)\alpha\beta} - N^{(1)\alpha} = I^{(1)\beta\alpha}\ddot{u}_\beta - I^{(2)\alpha\beta}\ddot{w}_{|\beta} + I^{(3)\beta\alpha}\ddot{\gamma}_\beta^0 - F^{(1)\alpha}$$

$$M_{|\alpha\beta}^{(2)\beta\alpha} + N^{(1)3} = I_{|\beta}^{(2)\alpha\beta}\ddot{u}_\alpha + I^{(2)\alpha\beta}\ddot{u}_{\alpha|\beta} + I^{(1)33}\ddot{w} - I_{|\beta}^{(4)\alpha\beta}\ddot{w}_{|\alpha} - I^{(4)\alpha\beta}\ddot{w}_{|\alpha\beta} + I_{|\beta}^{(6)\alpha\beta}\ddot{\gamma}_\alpha^0 + I^{(6)\alpha\beta}\ddot{\gamma}_{\alpha|\beta}^0 - P^3 - F^{(1)3} - F_{|\beta}^{(2)\beta} \quad (56)$$

$$M_{|\beta}^{(3)\alpha\beta} - N^{(2)\alpha} - N^{(3)\alpha} = I^{(3)\beta\alpha}\ddot{u}_\beta + I^{(5)\beta\alpha}\ddot{\gamma}_\beta^0 - I^{(6)\alpha\beta}\ddot{w}_{|\beta} - F^{(3)\alpha} \qquad (\alpha = 1, 2)$$

In the last expression, the generalized stresses are given by

$$\begin{aligned}
\left[N^{(1)^\alpha}, N^{(2)^\alpha}\right] &= \int_0^h \sigma^{\lambda\beta}\mu_\lambda^\nu\left[\mu^\alpha_{\nu|\beta}, h^\alpha_{\nu|\beta}\right]\mu \,\mathrm{d}x_3 \\
N^{(1)^3} &= \int_0^h \sigma^{\lambda\beta}\mu_\lambda^\nu b_{\nu\beta}\mu \,\mathrm{d}x_3 \\
\left[M^{(1)^{\alpha\beta}}, M^{(2)^{\alpha\beta}}, M^{(3)^{\alpha\beta}}\right] &= \int_0^h \sigma^{\lambda\beta}\mu_\lambda^\nu\left[\mu^\alpha_\nu, x_3\delta^\alpha_\nu, h^\alpha_\nu\right]\mu \,\mathrm{d}x_3 \\
N^{(3)^\alpha} &= \int_0^h \sigma^{\lambda 3}\left[\mu_\lambda^\nu h^\alpha_{\nu,3} + b_\lambda^\nu h^\alpha_\nu\right]\mu \,\mathrm{d}x_3
\end{aligned} \tag{57}$$

The generalized external forces are given by

$$\begin{aligned}
\left[F^{(1)^\alpha}, F^{(2)^\alpha}, F^{(3)^\alpha}\right] &= \int_0^h f^\nu \mu^\beta_\nu\left[\mu^\alpha_\lambda, \delta^\alpha_\lambda x_3, h^\alpha_\lambda\right]a^{\lambda\beta}\mu \,\mathrm{d}x_3 \\
F^{(1)^3} &= \int_0^h f^3\mu \,\mathrm{d}x_3 \\
\left[S^{(1)^\alpha}, S^{(2)^\alpha}, S^{(3)^\alpha}\right] &= \int_0^h s^{\nu\beta}\left[\mu^\alpha_\nu, h^\alpha_\nu, x_3\delta^\alpha_\nu\right]n_\beta\mu \,\mathrm{d}x_3 \\
S^{(1)^3} &= \int_0^h s^{3\beta}n_\beta\mu \,\mathrm{d}x_3, \qquad P^3 = -\mu_{(h)}p_h + p_0
\end{aligned} \tag{58}$$

and the inertia terms are given by

$$\begin{aligned}
&\left[I^{(1)^{\alpha\beta}}, I^{(2)^{\alpha\beta}}, I^{(3)^{\alpha\beta}}, I^{(4)^{\alpha\beta}}, I^{(5)^{\alpha\beta}}, I^{(6)^{\alpha\beta}}\right] \\
&\quad = \int_0^h \rho a^{\lambda\nu}\left[\mu^\beta_\nu\mu^\alpha_\lambda, x_3\delta^\beta_\lambda\mu^\alpha_\nu, \mu^\beta_\nu h^\alpha_\lambda, x_3^2\delta^\beta_\lambda\delta^\alpha_\nu, h^\beta_\nu h^\alpha_\lambda, \delta^\beta_\lambda h^\alpha_\nu\right]\mu \,\mathrm{d}x_3 \\
&I^{(1)^{33}} = \int_0^h \rho\mu \,\mathrm{d}x_3
\end{aligned} \tag{59}$$

The boundary conditions leading to a regular problem are

$$\begin{aligned}
M^{(1)^{\alpha\beta}}n_\beta &= S^{(1)^\alpha} \qquad \text{or} \qquad \delta u_\alpha = 0 \\
\left[\tfrac{1}{2}\left(M^{(2)^{\alpha\beta}} + M^{(2)^{\beta\alpha}}\right)_{|\alpha} + F^{(2)^\beta}\right]n_\beta &= S^{(1)^3} \qquad \text{or} \qquad \delta w = 0 \\
M^{(3)^{\alpha\beta}}n_\beta &= S^{(2)^\alpha} \qquad \text{or} \qquad \delta\gamma^0_\alpha = 0 \\
\tfrac{1}{2}\left[M^{(2)^{\alpha\beta}} + M^{(2)^{\beta\alpha}}\right]n_\beta &= S^{(3)^\alpha} \qquad \text{or} \qquad \delta w_{|\alpha} = 0
\end{aligned} \tag{60}$$

The displacement equations of motion are deduced from Eqs. (56–59), including the constitutive law given by Eq. (26).

V. Numerical Examples

To assess the accuracy of the present theory, the following problems, for which an exact three-dimensional elasticity solution is known, have been examined: first, the bending of a simply supported sandwich plate[16] and, second, the cylindrical bending of a simply supported cylindrical panel.[17]

The sensitivity of the model for a clamped, free-edge circular cylindrical panel is also examined.

A. Bending of a Simply Supported Sandwich Plate

Consider a rectangular, symmetric cross-ply (0/90/0 deg) laminated plate, simply supported on all edges, submitted to a doubly sinusoidal load. The layers have equal thickness, the fibers of the outer layers are oriented in the x_1 direction (0 deg), with those in the innermost layer in the x_2 direction (90 deg).

The plate is rectangular, $b = 3a$. The mechanical constants for each layer are

$$\begin{aligned}
E_L/E_T &= 25, \qquad G_{LT}/E_T = 0.5 \\
G_{TT}/E_T &= 0.2, \qquad \nu_{LT} = \nu_{TT} = 0.25
\end{aligned} \tag{61}$$

where L and T are the directions parallel and normal to the fibers, respectively, and ν_{LT} is Poisson's ratio measuring transverse strain under uniaxial stress parallel to the fibers. The plate is submitted to a transversly doubly sinusoidal distributed transverse loading

$$P = P_0\sin[(\pi/a)x_1]\sin[(\pi/b)x_2] \tag{62}$$

Table 1 contains nondimensionalized deflections and stresses such as

$$\begin{aligned}
\bar{w} &= \frac{E_T h^3 w(a/2, b/2)}{P_0 a^3}, \qquad \bar{\sigma}_{11} = \frac{E_T h^2\sigma_{11}(a/2, b/2, h)}{P_0 a^2} \\
\bar{\sigma}_{22} &= \frac{E_T h^2\sigma_{22}(a/2, b/2, 2h/3)}{P_0 a^2}, \qquad \bar{\sigma}_{23} = \frac{h\sigma_{23}(a/2, 0, h)}{P_0 a} \\
\bar{\sigma}_{13} &= \frac{h\sigma_{13}(0, b/2, h)}{P_0 a}, \qquad \bar{\sigma}_{12} = \frac{h^2\sigma_{12}(0, 0, h)}{P_0 a^2}
\end{aligned} \tag{63}$$

The results obtained from the present theory are compared with those obtained from the three-dimensional elasticity theory[16], the Béakou and Touratier[9] theory, the first-order shear deformation theory (FSDT),[18] and the refined Reddy theory.[18]

For this problem, the present theory yields results that are revealed to be closer to the exact three-dimensional elasticity solution than those just mentioned especially as concerns the shear stresses, for which a noticeable improvement appears.

Table 1 Nondimensionalized deflection and stresses in a laminated rectangular ($b = 3a$) simply supported plate under doubly sinusoidal transverse load

a/h	Theory	$\bar{w}$ $(a/2, b/2)$	$\bar{\sigma}_{11}$ $(a/2, b/2, h)$	$\bar{\sigma}_{22}$ $(a/2, b/2, 2h/3)$	$\bar{\sigma}_{12}$ $(0, 0, h)$	$\bar{\sigma}_{23}$ $(a/2, 0, h/2)$	$\bar{\sigma}_{13}$ $(0, b/2, h/2)$
4	Exact three dimensional	2.820	1.100	0.119	0.0281	0.0334	0.387
	Present	2.743	1.107	0.115	0.0287	0.0318	0.384
	Béakou–Touratier	2.729	1.213	0.106	0.0276	0.0298	0.378
	Reddy	2.641	1.036	0.103	0.0263	0.0348	0.272
	FSDT	2.363	0.613	0.093	0.0205	0.0308	0.188
10	Exact three dimensional	0.919	0.725	0.0435	0.0123	0.0152	0.420
	Present	0.917	0.720	0.0434	0.0121	0.0149	0.439
	Béakou–Touratier	0.918	0.733	0.0418	0.0122	0.0147	0.441
	Reddy	0.862	0.692	0.0398	0.0115	0.0170	0.286
	FSDT	0.803	0.621	0.0375	0.0105	0.0159	0.189
20	Exact three dimensional	0.6100	0.650	0.0299	0.0093	0.0119	0.434
	Present	0.6095	0.650	0.0298	0.0093	0.0115	0.448
	Béakou–Touratier	0.6095	0.650	0.0294	0.0093	0.0113	0.453
	Reddy	0.5937	0.624	0.0253	0.0083	0.0129	0.289
	FSDT	0.5060	0.623	0.0253	0.0083	0.0127	0.190

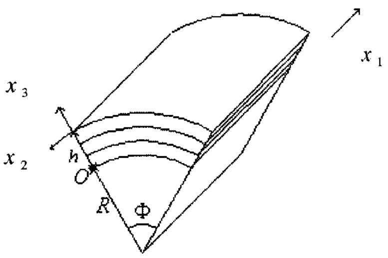

Fig. 2 Laminated circular cylindrical shell.

B. Cylindrical Bending of a Simply Supported Cylindrical Panel

The bending of a circular, cylindrical, laminated, symmetric cross-ply (0/90/0 deg) shell, of thickness h and inner radius R, with a central angle $\Phi = \pi/3$, infinite in the x_1 direction, so that it is in a state of plane strain with the (x_2, x_3) plane, is considered (Fig. 2). The polar angle coordinate will be denoted $\theta(-\Phi \le \theta \le 0)$.

The mechanical constants for each layer are identical to those given in Eq. (61). The panel is simply supported along its edges. Results obtained with our theory are, first, compared with the exact three-dimensional elasticity solution of the same problem as obtained by Ren.[17]

Distributions of dimensionless in-plane circumferential stress and transverse shear stress through the thickness are plotted, and the influence of curvature on the dissymmetry of the transverse shear stress through the thickness is shown.

The panel is submitted to a transverse loading

$$p = p_h \tag{64}$$

It is found that

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}, \qquad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/R \end{bmatrix}$$

$$[b_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & -R \end{bmatrix}, \qquad [b^{\alpha}_{\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & -1/R \end{bmatrix} \tag{65}$$

$$[\mu^{\alpha}_{\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & x_3/R \end{bmatrix}$$

According to Eq. (48), the displacement components of the shell are

$$U_1 = 0, \qquad U_2 = \mu_2^2 u_2 - x_3 w_{|2} + h_2^2 \gamma_2^0, \qquad U_3 = w \tag{66}$$

where

$$h_2^2 = \frac{h}{2R} x_3 + f(x_3) + \frac{\pi R}{2h}\left(2 + \frac{h}{R}\right) g(x_3) + \sum_{m=1}^{N-1}\left[-\frac{1}{2}x_3\left(1 + \frac{x_{3(m)}}{R}\right) - \frac{\pi R}{2h}\left(1 + \frac{x_{3(m)}}{R}\right) g(x_3) + \left(x_3 - x_{3(m)}\right) H\left(x_3 - x_{3(m)}\right)\right] a^2_{(m)^2} \tag{67}$$

and the associated strain components are

$$e_{22} = \mu_2^2\left[\mu_2^2 u_{2|2} - x_3 w_{|22} + h_2^2 \gamma^0_{2|2} - b_{22} w\right]$$
$$e_{23} = \tfrac{1}{2}\left[\mu_2^2 h^2_{2,3} + b_2^2 h_2^2\right]\gamma_2^0, \qquad e_{11} = e_{12} = e_{13} = 0 \tag{68}$$

From Eqs. (56–58) and Eq. (26), the following governing equations are deduced:

$$A^1 u_{2,22} + B^1 w_{,222} + C^1 \gamma^0_{2,22} + B'^1 w_{,2} = 0$$
$$A^2 u_{2,222} + B^2 w_{,2222} + C^2 \gamma^0_{2,222} + B'^2 w_{,22} + A^4 u_{2,2} + C^4 \gamma^0_{2,2} + B^4 w = [1 + (h/R)]p \tag{69}$$
$$A^3 u_{2,22} + B^3 w_{,222} + C^3 \gamma^0_{2,22} - C^5 \gamma_2^0 + B'^4 w_{,2} = 0$$

where the values of the constants $A^1, B^1, \ldots,$ are given in Appendix A.

From Eq. (60), the boundary conditions are

$$w(\theta = 0) = w(\theta = -\Phi) = 0$$
$$M^{(1)^{22}}(\theta = 0) = M^{(1)^{22}}(\theta = -\Phi) = 0 \tag{70}$$
$$M^{(3)^{22}}(\theta = 0) = M^{(3)^{22}}(\theta = -\Phi) = 0$$

If the transverse load can be expressed as

$$p = \sum_{n=1}^{\infty} P_0^{(n)} \sin(\lambda x_2) \tag{71}$$

then, the solutions of Eqs. (69) are obtained as

$$u_2 = \sum_{n=1}^{\infty} A_2^{(n)} \cos(\lambda x_2), \qquad w = \sum_{n=1}^{\infty} B^{(n)} \sin(\lambda x_2)$$
$$\gamma_2^0 = \sum_{n=1}^{\infty} C_2^{(n)} \cos(\lambda x_2) \tag{72}$$

where

$$\lambda = n\pi/\Phi \tag{73}$$

and $A_2^{(n)}$, $B^{(n)}$, and $C_2^{(n)}$ are constants to be determined from the following linear algebraic equations:

$$\boldsymbol{K}\begin{bmatrix} A_2^{(n)} \\ B^{(n)} \\ C_2^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ [1 + (h/R)]P_0^{(n)} \\ 0 \end{bmatrix} \tag{74}$$

where the coefficients of the stiffness matrix $\boldsymbol{K}$ are given in Appendix B.

Using Eqs. (66–68) and Eq. (26), the displacement, strain, and stress components of the shell can be calculated.

As a special case, numerical computations are performed for

$$p = p_h = P_0 \sin(\pi x_2/\Phi) \tag{75}$$

The ratio

$$s = \frac{R + 0.5h}{h} \tag{76}$$

which measures the relative thickness of the laminated shell, will be used in the following.

Distributions through the thickness of the shell of dimensionless in-plane circumferential stress and dimensionless transverse shear stress are plotted. The good agreement of the results yielded by the proposed model with the exact three-dimensional elasticity solution can be seen in Table 2, where a comparison with results obtained by He[3] and with the classical thin shell theory is made.

Figure 3 shows, for $s = 10$, the variations through the thickness of the shell of dimensionless circumferential shear stress

$$\bar{\sigma}_{22} = \frac{\sigma_{22}(\theta = -\pi/3)}{P_0 s^2} \tag{77}$$

Figure 4 shows, for $s = 10$, the variations through the thickness of the shell of dimensionless transverse shear stress

$$\bar{\sigma}_{23} = \frac{\sigma_{23}(\theta = -\pi/3)}{P_0 s} \tag{78}$$

From Table 2 and Figs. 3 and 4, it may be noted that the proposed model yields results closer to the three-dimensional solution than previous theories. The distribution of the transverse shear stress through the thickness of the shell is nearly parabolic and satisfies both compatibility conditions at layer interfaces and on the bounding surfaces. It clearly appears that no shear-correction factors are required.

Table 2 Nondimensionalized deflection and stresses in a laminated circular cylindrical simply supported panel under sinusoidal transverse load

s	Theory	$\bar{w}$ $(0, \Phi/2)$	Error, %	$\bar{\sigma}_{22}$ $(0, \Phi/2) - (h, \Phi/2)$	$\bar{\sigma}_{23}$ $(h/2, 0)$
2	Exact three dimensional	1.436		2.463 −3.467	0.394
	Present	1.538	7	2.434 −3.26	0.358
	He	1.59	10.72		0.480
	CST	0.0799	106	0.686 −0.870	——
4	Exact three dimensional	0.457		1.367 −1.772	0.476
	Present	0.494	7	1.438 −1.819	0.470
	He	0.500	9.4		0.396
	CST	0.0781	82.91	0.732 −0.824	——
10	Exact three dimensional	0.144		0.897 −0.995	0.525
	Present	0.145	4	0.97 −1	0.519
	He	0.152	5.5		0.590
	CST	0.0777	46	0.759 −0.796	——
50	Exact three dimensional	0.0808		0.782 −0.798	0.526
	Present	0.0802	0.69	0.787 −0.794	0.519
	He	0.0800	0.99		0.520
	CST	0.0776	3.96	0.774 −0.792	——
100	Exact three dimensional	0.0787		0.781 −0.786	0.523
	Present	0.0783	0.5	0.777 −0.780	0.520
	He	0.0780	0.89		0.520
	CST	0.0776	1.39	0.776 −0.779	——

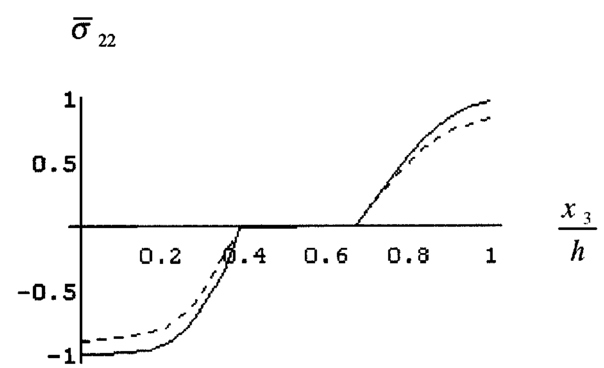

Fig. 3 Distribution of nondimensionalized circumferential stress through the thickness of the shell at $\theta = -(\pi/3)$ for $s = 10$: ——, present theory, and – – –, exact solution.

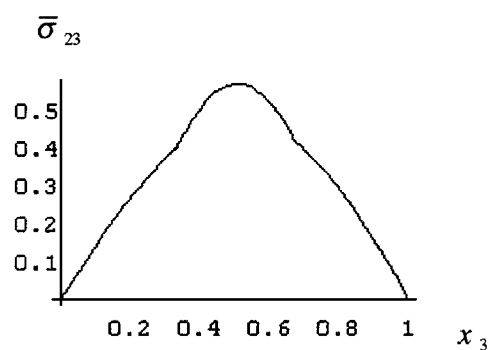

Fig. 4 Distribution of nondimensionalized transverse shear stress through the thickness of the shell at $\theta = -(\pi/3)$ for $s = 10$.

C. Clamped Free-Edge Panel

The sensitivity of the theory to edge effects for a clamped, free-edge circular cylindrical panel is examined. The mechanical and geometrical properties of the considered panel are the same as those described in Sec. V.B. The panel is submitted to a simple sinusoidal load,

$$p = p_h = P_0 \sin(\lambda x_2) \tag{79}$$

where

$$\lambda = \pi / \Phi \tag{80}$$

Two ways are known to take into account edge effects: 1) the correction of the elementary theories near the edge zone (see, among others, Refs. 19 and 20) and 2) the development of higher models (see Refs. 21 and 22). Our approach is of the second kind. In the absence of an exact three-dimensional elasticity solution for the considered problem, the numerical reference solution for the following analysis is based on a three-dimensional solid finite element computation.

The boundary conditions are, for the clamped edge

$$\boldsymbol{U}(\theta = 0) = 0 \tag{81}$$

and for the free edge

$$\begin{aligned} M^{(1)^{22}}(\theta = -\Phi) &= 0, & M^{(2)^{22}}(\theta = -\Phi) &= 0 \\ M^{(3)^{22}}(\theta = -\Phi) &= 0, & N^{(1)^{3}}(\theta = -\Phi) &= 0 \end{aligned} \tag{82}$$

As

$$\boldsymbol{U} = \begin{cases} U_1 = 0 \\ U_2 = \mu_2^2 u_2 - x_3 w_{|2} + h_2^2 \gamma_2^0 \\ U_3 = w \end{cases} \tag{83}$$

Eq. (81) results in

$$u_2(\theta = 0) = 0, \qquad w_{,2}(\theta = 0) = 0$$
$$\gamma_2^0(\theta = 0) = 0, \qquad w(\theta = 0) = 0 \tag{84}$$

Because of the boundary conditions obtained via Hamilton's principle, eight equations are obtained, which leads to the determination of the eight independent constants of integration.

The solution, in term of generalized displacements, is obtained by combining the general solution of the corresponding homogeneous system with a particular solution of the complete system. The problem considered requires solving the following governing equations from Eqs. (56):

$$A^1 u_{2,22} + B^1 w_{,222} + C^1 \gamma_{2,22}^0 + B'^1 w_{,2} = 0 \tag{85a}$$

$$A^2 u_{2,222} + B^2 w_{,2222} + C^2 \gamma_{2,222}^0 + B'^2 w_{,22} + A^4 u_{2,2}$$
$$+ C^4 \gamma_{2,2}^0 + B^4 w = [1 + (h/R)]p \tag{85b}$$

$$A^3 u_{2,22} + B^3 w_{,222} + C^3 \gamma_{2,22}^0 - C^5 \gamma_2^0 + B'^4 w_{,2} = 0 \tag{85c}$$

where A^1, $B^1, \ldots,$ are the same constants as in Eq. (69).

The solution of the homogeneous system is searched as

$$u_2 = U_0\, e^{sx_2}, \qquad w = W_0\, e^{sx_2}, \qquad \gamma_2^0 = \Gamma_0\, e^{sx_2} \tag{86}$$

Substituting Eq. (86) into the homogeneous system corresponding to Eq. (85), three linear equations in terms of U_0, W_0, and Γ_0 are obtained. For a nontrivial solution, the determinant of the coefficient matrix must vanish, resulting in a polynomial equation in s.

An eighth-order equation in s is obtained. Beside the $s = 0$ double root, there are two purely real and four complex roots. They will be denoted s_1, s_2, s_3, s_4, s_5, and s_6, respectively.

A particular solution of Eq. (85) is given by

$$u_2 = 0, \qquad w = -\left[(\mu_h P_0)/B^4\right], \qquad \gamma_2^0 = 0 \tag{87}$$

Therefore, the final solution is obtained as

$$u_2 = \alpha_1\, e^{s_1 x_2} + \cdots + \alpha_6\, e^{s_6 x_2} + P_1$$
$$w = \beta_1\, e^{s_1 x_2} + \cdots + \beta_6\, e^{s_6 x_2} + P - \left(\mu_h P_0 / B^4\right) \tag{88}$$
$$\gamma_2^0 = \gamma_1\, e^{s_1 x_2} + \cdots + \gamma_6\, e^{s_6 x_2} + P_2$$

where $\alpha_1, \ldots, \alpha_6$, $\beta_1, \ldots, \beta_6$, and $\gamma_1, \ldots, \gamma_6$ are constants of integration and P, P_1, and P_2 polynomial functions of the first degree in x_2

$$P = a_1 x_2 + a_0, \qquad P_1 = b_1 x_2 + b_0, \qquad P_2 = c_1 x_2 + c_0 \tag{89}$$

whose presence is due to the double root $s = 0$.

Hence, 24 constants ($\alpha_1, \ldots, \alpha_6$, $\beta_1, \ldots, \beta_6$, $\gamma_1, \ldots, \gamma_6$, a_1, a_0, b_1, b_0, c_1, and c_0) have to be determined. A 24×24 system of equations is, therefore, required. This system is obtained in the following way. First, two equations are choosen in the differential system (85), where u_2, w, and γ_2^0 are replaced by their expression obtained in Eq. (88). The exponential functions $e^{s_1 x_2}, \ldots, e^{s_6 x_2}$ and the polynomial functions being independent, identification leads to 16 linear equations in $\alpha_1, \ldots, \alpha_6$, $\beta_1, \ldots, \beta_6$, $\gamma_1, \ldots, \gamma_6$, a_1, a_0, b_1, b_0, c_1, and c_0. Second, Eq. (88) is then used in the boundary conditions (81) and (82), leading to 8 linear equations in terms of $\alpha_1, \ldots, \alpha_6$, $\beta_1, \ldots, \beta_6$, $\gamma_1, \ldots, \gamma_6$, a_1, a_0, b_1, b_0, c_1, and c_0. The required system of 24 linear equations in terms of $\alpha_1, \ldots, \alpha_6$, $\beta_1, \ldots, \beta_6$, $\gamma_1, \ldots, \gamma_6$, a_1, a_0, b_1, b_0, c_1, and c_0 is thus obtained, allowing the determination of those constants.

With Eqs. (66–68) and the constitutive law (26), the displacement, strain, and stress components of the shell can be calculated. Results obtained with the model have been compared with three-dimensional 20-node brick elements (4500 elements, which correspond to 63,777 degrees of freedom) and two-dimensional eight-node Reissner–Mindlin finite elements (5000 elements, which correspond to 91,806 degrees of freedom) computations, as shown in Figs. 5 and 6. In terms of finite element nodal constraints, the clamped support is simulated by setting the components of the displacement vector equal to zero. The length of the cylinder has been taken equal to $a = 10h$.

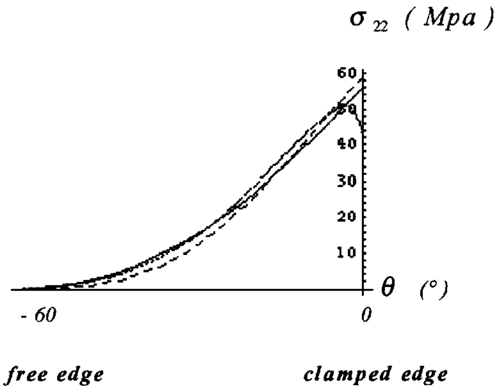

Fig. 5 Distribution of σ_{22} along the circumference of the shell at $x_3 = 2h/3$ for the clamped free-edge panel ($s = 10$): ——, present theory; – – –, three-dimensional 20-node solid finite element model from ABAQUS; and — —, two-dimensional 8-node finite element model from ABAQUS.

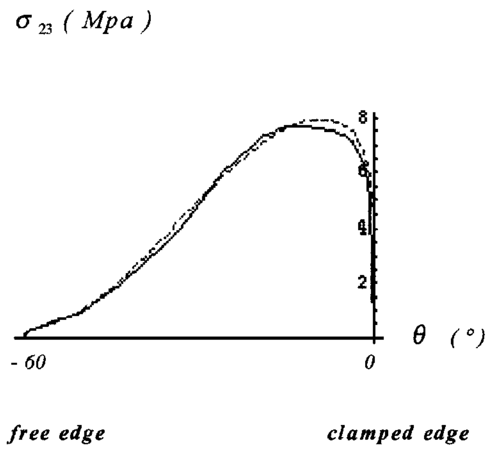

Fig. 6 Distribution of σ_{23} through the circumference of the shell at $x_3 = 2h/3$ for the clamped free-edge panel ($s = 10$): ——, present theory, and – – –, three-dimensional 20-node solid finite element model from ABAQUS.

From Fig. 5 and Table 3, the following remarks can be made. First, the boundary condition (nullity of the in-plane stress σ_{22} at the free edge) is satisfied; then, the circumferential stress is not a maximum at the clamped edge, but in a region close to it. To assess the accuracy of the proposed model, results from a two-dimensional shell finite element computation have been considered. Results obtained with the present theory are seen to be closer to those given by the three-dimensional solid finite element (Reissner–Mindlin type) model than those of the two-dimensional shell finite element model. Far from the clamped edge, results are almost identical. They begin to differ as one approaches the clamped edge. Note that the two-dimensional shell finite-element yields less accurate results than the present shell theory. The edge effects, which lead to rapid changes in stress profiles, are confined in a region close to the clamped edge with a length which is of the order of the thickness of the shell. Similar results have already been observed in Ref. 23.

From Fig. 6 and Table 4, the following remarks can be made. In the vicinity of the free edge, results obtained from the present theory are close to those obtained with the three-dimensional solid finite element computations. Results tend to separate as one approaches the clamped edge, especially as concerns the maximum value of the transverse shear stress, located in a region close to the clamped edge. As with other refined theories, our theory gives an obviously incorrect result at the clamped edge, due to the imposed boundary condition $\gamma_2^0(x_2 = 0) = 0$ [see Eq. (84)]. Refined theories based on Eq. (12) are, of course, more sensitive to edge effects than first-order theories.[12]

Table 3 Values of the circumferential stress σ_{22} along the circumference of the shell at $x_3 = 2h/3$ for the clamped free-edge panel ($s = 10$)

θ, deg	σ_{22}, Mpa ($x_3 = 2h/3$) Present theory	σ_{22}, Mpa ($x_3 = 2h/3$) Three-dimensional solid finite element model	σ_{22}, Mpa ($x_3 = 2h/3$) Two-dimensional solid finite element model
0	56.19	42.91	59.26
−0.63	55.13	45.61	58.18
−1.26	54.08	48.33	57.09
−1.89	53.03	49.62	56.00
−2.53	51.99	50.63	54.91
−3.16	50.96	50.54	53.81
−3.79	49.94	50.73	52.71
−4.42	48.93	50.36	51.06
−5.05	47.93	49.96	50.50
−5.68	46.94	49.31	49.39
−6.315	45.96	48.66	53.81
−6.324	44.98	47.89	52.71
−7.56	44.02	47.12	——
−8.21	43.06	46.28	——
−8.84	42.12	45.44	——
−9.47	41.18	44.55	——
−12.63	36.65	39.9	37.41
−18.95	28.32	30.23	27.29
−25.26	20.99	21.35	18.49
−31.58	14.71	13.98	11.37
−37.89	9.49	8.35	6.12
−44.21	5.39	4.44	2.68
−50.53	2.42	2.03	0.8132
−59.53	0.61	0.71	9.72E−02
−60	0	0	1.20E−05

Table 4 Values of the transverse shear-stress σ_{23} along the circumference of the shell at $x_3 = 2h/3$ for the clamped free-edge panel ($s = 10$)

θ, deg	σ_{23}, Mpa ($x_3 = 2h/3$) Present theory	σ_{23}, Mpa ($x_3 = 2h/3$) Three-dimensional solid finite element model
0	0	4.656
−0.63	5.042	5.548
−1.26	6.014	6.441
−1.89	6.292	6.812
−2.53	6.702	7.159
−3.16	6.936	7.303
−3.79	7.155	7.522
−4.42	7.247	7.614
−5.05	7.344	7.711
−5.68	7.408	7.775
−6.324	7.481	7.848
−6.947	7.511	7.879
−7.56	7.549	7.917
−8.21	7.567	7.934
−8.84	7.583	7.951
−10.105	7.642	7.945
−11.368	7.662	7.926
−12.63	7.672	7.835
−13.895	7.682	7.738
−15.158	7.697	7.615
−16.421	7.643	7.471
−17.684	7.479	7.306
−18.95	7.389	7.124
−25.26	6.175	6.001
−31.58	4.385	4.659
−37.89	3.073	3.265
−44.21	1.843	1.980
−50.53	0.944	0.947
−59.53	0.199	0.280
−60	0	0

VI. Conclusion

A new refined bidimensional shell theory, which allows the compatibility conditions for displacements and stresses at layer interfaces to be exactly satisfied, is proposed. This theory, which keeps only five independent generalized displacements, also takes into account refinements of the shear and membrane terms, by means of trigonometric functions. The accuracy of the model is assessed by applying the theory to problems for which an exact three-dimensional elasticity solution is known. Through comparison with results given by previous theories (first-order shear deformation theory, Reddy theory, Béakou–Touratier theory, and He theory) and with, of course, the exact solution, it is shown that the present model lightly improves accuracy.

The sensitivity of the proposed model to edge effects for a clamped, free-edge panel is also examined. In common with other refined theories, the present theory appears to be more sensitive to edge effects than the first-order shear-deformation theory. This new theory has also been extended to dynamics, in another paper.

Appendix A: Coefficients

The coefficients appearing in Eqs. (73) are given by

$$A^1 = \int_0^h \frac{C_{2222}}{\alpha_2}\left(\mu_2^2\right)^4 \mu \,\mathrm{d}x_3, \qquad A^3 = \int_0^h \frac{C_{2222}}{\alpha_2}\mu_2^3 h_2^2 \mu \,\mathrm{d}x_3$$

$$B^1 = -\int_0^h \frac{C_{2222}}{\alpha_2} x_3 \left(\mu_2^2\right)^3 \mu \,\mathrm{d}x_3$$

$$B^3 = -\int_0^h \frac{C_{2222}}{\alpha_2} x_3 \mu_2^2 h_2^2 \mu \,\mathrm{d}x_3$$

$$C^1 = \int_0^h \frac{C_{2222}}{\alpha_2} h_2^2 \left(\mu_2^2\right)^3 \mu \,\mathrm{d}x_3, \qquad C^3 = \int_0^h \frac{C_{2222}}{\alpha_2} \mu_2^2 \left(h_2^2\right)^2 \mu \,\mathrm{d}x_3$$

$$B'^1 = -\int_0^h \frac{C_{2222}}{\alpha_2} b_{22} \left(\mu_2^2\right)^3 \mu \,\mathrm{d}x_3$$

$$B'^4 = -\int_0^h \frac{C_{2222}}{\alpha_2} b_{22} \mu_2^2 h_2^2 \mu \,\mathrm{d}x_3$$

$$A^2 = \int_0^h \frac{C_{2222}}{(\alpha_2)^2} \left(\mu_2^2\right)^3 x_3 \mu \,\mathrm{d}x_3, \qquad A^4 = \int_0^h C_{2222} \left(\mu_2^2\right)^2 b_{22} \mu \,\mathrm{d}x_3$$

$$B^2 = -\int_0^h \frac{C_{2222}}{(\alpha_2)^2} x_3 \left(\mu_2^2\right)^2 x_3 \mu \,\mathrm{d}x_3$$

$$B'^4 = -\int_0^h C_{2222} x_3 \mu_2^2 b_{22} \mu \,\mathrm{d}x_3$$

$$C^2 = \int_0^h \frac{C_{2222}}{(\alpha_2)^2} h_2^2 \left(\mu_2^2\right)^2 x_3 \mu \,\mathrm{d}x_3, \qquad C^4 = \int_0^h C_{2222} h_2^2 \mu_2^2 b_{22} \mu \,\mathrm{d}x_3$$

$$B''^2 = -\int_0^h \frac{C_{2222}}{(\alpha_2)^2} b_{22}\left(\mu_2^2\right)^2 x_3 \mu \, dx_3$$

$$B^4 = -\int_0^h C_{2222}\mu_2^2 (b_{22})^2 \mu \, dx_3$$

$$C^5 = \int_0^h C_{2323}\left[\mu_2^2 h_{2,3}^2 + b_2^2 h_2^2\right]^2 \mu \, dx_3, \qquad B'^2 = B''^2 + B'^4$$

Numerical integration is performed with a Gauss rule.

Appendix B: Elements

The elements of the stiffness matrix $\boldsymbol{K}$ appearing in Eq. (78) are given, where K_{ij} is an element in the ith row, jth column,

$$K_{11} = -\lambda^2 \Lambda_u^{(1)}, \qquad K_{21} = \left(\lambda^3 \Lambda_u^{(2)} - \lambda T_u^{(1)}\right), \qquad K_{31} = -\lambda^2 \Lambda_u^{(3)}$$

$$K_{12} = \lambda \Lambda_w^{(1)}, \qquad K_{22} = \left(-\lambda^2 \Lambda_w^{(2)} + T_w^{(1)}\right), \qquad K_{32} = \lambda \Lambda_w^{(3)}$$

$$K_{13} = \lambda^2 \Lambda_\gamma^{(1)}, \qquad K_{23} = \left[\lambda^3 \Lambda_\gamma^{(2)} - \lambda T_\gamma^{(1)}\right]$$

$$K_{33} = \left[\lambda^2 \Lambda_\gamma^{(3)} + T_\gamma^{(3)}\right]$$

where

$$\Lambda_u^{(1)} = -\int_0^h a^{22} C_{2222}\left(\mu_2^2\right)^4 \mu \, dx_3$$

$$\Lambda_u^{(3)} = -\int_0^h C_{2222} a^{22}\left(\mu_2^2\right)^3 h_2^2 \mu \, dx_3$$

$$\Lambda_w^{(1)} = \int_0^h \frac{1}{\alpha_2} C_{2222}\left(a^{22} x_3 \lambda^2 - b_{22}\right)\left(\mu_2^2\right)^3 \mu \, dx_3$$

$$\Lambda_w^{(3)} = \int_0^h \frac{1}{\alpha_2} C_{2222}\left(a^{22} \lambda^2 x_3 - b_{22}\right)\left(\mu_2^2\right)^2 h_2^2 \mu \, dx_3$$

$$\Lambda_\gamma^{(1)} = -\int_0^h a^{22} C_{2222} h_2^2 \left(\mu_2^2\right)^3 \mu \, dx_3$$

$$\Lambda_\gamma^{(3)} = -\int_0^h C_{2222} a^{22} h_2^2 \left(\mu_2^2\right)^2 h_2^2 \mu \, dx_3$$

$$\Lambda_u^{(2)} = -\int_0^h \frac{1}{\alpha_2} C_{2222} a^{22}\left(\mu_2^2\right)^3 x_3 \mu \, dx_3$$

$$T_u^{(1)} = -\int_0^h \frac{1}{\alpha_2} C_{2222}\left(\mu_2^2\right)^2 b_{22} \mu \, dx_3$$

$$\Lambda_w^{(2)} = \int_0^h C_{2222} a^{22}\left(a^{22} \lambda^2 x_3 - b_{22}\right)\left(\mu_2^2\right)^2 x_3 \mu \, dx_3$$

$$T_w^{(1)} = \int_0^h C_{2222}\left(a^{22} x_3 - b_{22}\right)\mu_2^2 b_{22} \mu \, dx_3$$

$$\Lambda_\gamma^{(2)} = -\int_0^h \frac{1}{\alpha_2} C_{2222} a^{22} h_2^2 \left(\mu_2^2\right)^2 x_3 \mu \, dx_3$$

$$T_\gamma^{(1)} = -\int_0^h \frac{1}{\alpha_2} C_{2222} h_2^2 \mu_2^2 b_{22} \mu \, dx_3$$

$$T_\gamma^{(3)} = \left[\int_0^h C_{2323}\left[\mu_2^2 h_{2,3}^2 + b_2^2 h_2^2\right]^2 \mu \, dx_3\right]$$

Numerical integration is performed with a Gauss rule.

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